International Journal of Applied Mathematics
& Statistical Sciences (IJAMSS)
ISSN(P): 2319-3972; ISSN(E): 2319-3980
Vol. 5, Issue 4, Jun – Jul 2016; 9-22

International Academy of Science, Engineering and Technology
Connecting Researchers; Nurturing Innovations

S-ITERATION FOR GENERAL QUASI MULTI VALUED CONTRACTION MAPPINGS

SALWA SALMAN ABED & RANA FADHIL ABBAS

Def of Math College of Education Ibn Al-Haitham for Pure Sciences, University of Baghdad

ABSTRACT

In this paper, the convergence of s-iteration sequence for general quasi contraction multi valued mappings is studied, where its rate of convergence is compared with Picard-Mann iteration sequence and show that s-iteration is faster than Picard-Mann iteration. Finally, a numerical example is given.

KEYWORDS: Fixed Point, General Quasi Multivalued Contraction Mappings, Iteration Processes, Normed Spaces

1. INTRODUCTION

Let X be a Banach space the classical Banach's contraction, see [22] shows that the Picard iterationP_n .

$P_n : x_{n+1}= Fx_n n \geq 0$, where $x_0 \in X$

Converges to unique fixed point z of contraction mapping$F: X \to X$, i.e.$\exists\ \alpha \in (1,0)$ such that

$\| F$x-Fy$\| \leq \alpha \| x - y \|$, for all x, y in X (1.1)

With priori error estimates

$\| x_{n-}z \| \leq \frac{\alpha^n}{1-\alpha} \| x_0 - x_1 \|$ n=0, 1, 2

and posteriori error estimates

$\| x_{n-}z \| \leq \frac{\alpha}{1-\alpha} \| x_{n-1} - x_n \|$n=0, 1, 2

its rate of convergence is obtained by

$\| x_n - z \| \leq \alpha \| x_n - z \| \leq \alpha^n . \| x_0 - z \|$.

For various generalizations of Branch's contraction mappings (1.1), much attention has been given to get many convergence results forP_n iteration such as, for Kannan's mappings[3], Chatterjea's mappings [4]and Zamfirescu mappings (or Z-operator) [5]which is a generalization of the independence mappings Banach's, Kannan's and Chatterjea's [12] contractive mappings (on compact normal space). For multi-valued contraction the argument of the proof of [theorem 5, 2] included a proof of the convergence of P_n iteration

$x_0 \in X$, $x_{n+1} \in Fx_n$ n=1, 2 (1.2)

to some fixed point of F, where Fx is nonempty closed and bounded subset of X.

Ciric [1] proved thatP_n iteration converges to the unique fixed point of a quasi- contraction multi-valued mappings. and gave a formula to posteriori error estimation. Moreover, Dung, el. at [20] gave a more general theorem which covered all previous cases in [theorem 3, 1], where the convergence of $\langle x_{n+1}\rangle$in (1.2) and posteriori error estimates for quasi-contraction multi-valued mappings are discussed.

On the other hand, other types of iteration are appeared which are convergence to a fixed point of quasi contraction mappings, like Mann iteration [13], Ishikawa iteration[14], s-iteration [15], two-step Mann iteration [16], Picard-Mann iteration [17], Picard-S iteration [18]. For the contraction mappings and their generalizations, many results are appeared which are included the convergence of various types of iteration processes such as [7], [8], [9],[19]. and the equivalence between some of these types of iterations, such as, in [11] Mann and Ishikawa iteration are equivalent when dealing with z-operators. Babu and Prasad [6] showed that Mann iteration converges faster than Ishikawa iteration for the class of z-operators. Also, in view of [7], the Picard iteration converges faster than Ishikawa iteration for these same class of mappings. In [15] that s-iteration converges faster than Mann iteration and Ishikawa iteration for z-operators. Also, there are some results showing that Picard iteration faster than Mann and Ishikawa iteration for quasi contraction mapping see [6], [1]

Here, the convergence of s - iteration sequence to fixed point is proved for general quasi contraction multi-valued mappings (shortly, g. q. m. c-mappings). And the equivalence of convergence between s-iteration andP_n -Mann iteration, the s-iteration converges faster thanP_n -Mann iteration is studied.

2. PRELIMINARIES

Let X be a Banach space and $F: X \to 2^X$ be a multivalued mapping,$x_0 \in X$ and$\langle\alpha_n\rangle$,$\langle\beta_n\rangle$be a sequences of real numbers in (0,1). In the following, we state some types of iteration processes for F at x_0:

- The Mann iteration of FM_n is defined by the sequence $\langle x_n\rangle$:

$$\begin{cases} x_0 \in X \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n\mu_n \end{cases} \text{for n} \geq 0 \tag{2.1}$$

Where $\mu_n \in Fx_n$, $\xi_n \in Fx_n$

- The Picard Mann iteration of FP_nM_n is defined by the sequence$\langle x_n\rangle$:

$$\begin{cases} x_{n+1} = \xi_n \\ y_n = (1-\alpha_n)x_n + \alpha_n\mu_n \end{cases} \text{for n} \geq 0 \tag{2.2}$$

Where $\mu_n \in Fx_n$, $\xi_n \in Fx_n$

- The 2- step Mann iteration of $F2M_n$ is defined by

The sequence$\langle x_n\rangle$:

$$\begin{cases} x_0 \in X \\ x_{n=1} = (1-\alpha_n)y_n + \alpha_n\,\xi_n \text{ for n} \geq 0 \\ y_n = (1-\beta_n)x_n + \beta_n\mu_n \end{cases} \tag{2.3}$$

Where $\mu_n \in Fx_n$, $\xi_n \in Fx_n$

- The Ishikawa iteration of FI_n is defined by

The sequence$\langle x_n\rangle$:

$$\begin{cases} x_0 \in X \\ x_{n=1} = (1-\alpha_n)x_n + \alpha_n \xi_n \text{ for n} \geq 0 \\ y_n = (1-\beta_n)x_n + \beta_n \mu_n \end{cases} \tag{2.4}$$

Where $\mu_n \in Fx_n, \xi_n \in Fx_n$

The s- iteration of FS_n is defined by

The sequence $\langle x_n \rangle$:

$$\begin{cases} x_0 \in X \\ x_{n=1} = (1-\alpha_n)\mu_n + \alpha_n \xi_n \text{ for n} \geq 0 \\ y_n = (1-\beta_n)x_n + \beta_n \mu_n \end{cases} \tag{2.5}$$

where $\mu_n \in Fx_n, \xi_n \in Fx_n$

Definition (2.1): [8]**:** Let $\langle a_n \rangle$ and $\langle b_n \rangle$ be two sequences of real numbers that converge to a and b respectively, and assume there exists

$l = \lim_{n\to\infty} \frac{|a_n - a|}{|b_n - b|}$, then if l =0, then we say that $\langle a_n \rangle$ converges faster to a than $\langle b_n \rangle$ to b.

Definition (2.2): for any two nonempty subsets M and N of X the Hausdorff distance is

$$D(M,N) = \max\{sup_{x\in M}\, d(x,N), sup_{y\in N}\, d(y,M)\}$$

Where $d(x,N) = \inf\{d(x,y): y \in N\}$

Definition (2.3): [1]: let $x_0 \in X$, an orbit of F at x_0 is a sequence $\{x_n : x_n \in Fx_{n-1}, n \in \mathbb{N}\}$

A space X is called to be F-orbitally complete if every Cauchy sequence

Which is a sub sequence of an orbit of F at x for some $x \in X$, converge in X

Definition (2.4): [20]: Let $F: X \to X$ be a mapping on metric space X. The mapping F is said to be a (g. q. m. c-mappings)iff there exists

$q \in [0,1)$ Such that for all $x, y \in X$,

$$D(Fx,Fy) \leq q\max\{d(x,y), d(x,Fx), d(y,Fy), d(x,Fy), d(y,Fx) \tag{2.6}$$

$$d(F^2x,x), d(F^2x,Fx), d(F^2x,y), d(F^2x,Fy)\}$$

Theorem (2.5): [Theorem (3.4), 20]: let (X,d) be ametric space and

F:X→ $CB(X)$ be ag. q. m. c-mapping If X is F- orbitally complete. Then

- F has a unique fixed point z in X and $Fz = \{z\}$
- for each $x_0 \in X$ there exists an orbit $\langle x_n \rangle$ of F at x_0 such that $\lim_{n\to\infty} x_n = z$ for all $x \in X$ and

$d(x_n, z) \leq \frac{(q^{1-a})^n}{1-q^{1-a}}\, d(x_0, x_1)$ For all $n \in N$, where $a < 1$ is any fixed positive number

As special cases of contraction condition (2.6) are, for x, y in X, Banach` s multivalued contraction condition is

$D(Fx, Fy) \leq ad(x, y)$ Where $0 \leq a < 1$ (2.7)

Kannan` s multivalued contraction condition is

$D(Fx, Fy) \leq b[d(x, Fx) + d(y, Fy)]$ Where $0 \leq b \leq 0.5$ (2.8)

Chatterjea` s multivalued contraction condition is

$D(Fx, Fy) \leq c[d(x, Fy) + d(y, Fx)]$ Where $0 \leq c \leq 0.5$ (2.9)

z-multivalued contraction condition (z-operator)

(z1) $D(Fx, Fy) \leq ad(x, y)$

(z2) $D(Fx, Fy) \leq b[d(x, Fx) + d(y, Fy)]$

(z3) $D(Fx, Fy) \leq c[d(x, Fy) + d(y, Fx)]$

where $0 \leq a < 1, 0 \leq b < 0.5, 0 \leq c < 0.5$ (2.10)

multivalued quasi - contraction (Ciric contraction) is

$D(Fx, Fy) \leq qmax\{d(x, y), d(x, Fx), d(y, Fy), d(x, Fy), d(y, Fx)$ (2.11)

It is Know that the contractions (2.7), (2.8) and (2.9) are independent [21] and the (2.10) is a generalization of them [8`]. Dung and el at gave the following example to show that the contraction a g. q. m. c-mappings is a generalization of (2.11)

Example (2.3)

Let X = {1,2,3,4,5} with d defined as:

$$d(x, y) = \begin{cases} 0 \; if \; x = y \\ 2 \; if \; (x, y) & \in \{(1,4), (1,5), (4,1), (5,1)\} \\ 1 \; otherwise \end{cases}$$

Let $F: X \rightarrow X$ be defined by

$$F1 = F2 = F3 = 1, F4 = 2, F5 = 3$$

F is not quasi-contraction for x =4 and y=5 because there is no a nonnegative number $q < 1$ satisfying the equation (2.6). However,F is generalized quasi-contraction since the (2.6) hold for some $q \in [0.5,1)$, for all $x, y \in X$.

3. MAIN RESULTS

we start with following theorem:

Theorem (3.1): let $\emptyset \neq$ M be a convex subset of a Banach space X and$F: M \rightarrow CB(M)$ is g.q.m.c-mappings. let $x_0 \in$ M and $\langle x_n \rangle$ be$S_n$iteration with $\sum_{n=0}^{\infty} \alpha_n = \infty, \lim_{n\rightarrow\infty} \beta_n = 0$. then $\langle x_n \rangle$converges strongly to a fixed point of F.

To prove we need the following lemma:

Lemma (3.2): let$X$, M, $F$and $\langle x_n\rangle$ as in theorem (3.1) then the sequences

$\langle x_n\rangle$,$\langle y_n\rangle$,$\langle \mu_n\rangle$,$\langle \xi_n\rangle$ are bounded where $x_n, y_n, \mu_n, \xi_n$is defined in (2.5)

Proof: for each n≥ 0, define

$A_n = \{\langle x_i\rangle \cup \langle y_i\rangle \cup \langle \mu_i\rangle \cup \langle \xi_i\rangle$, where$0 \le i \le n\}$

and $c_n = diam(A_n)$

d_n =max $\{sup_{0\le i\le n}\|x_0 - \mu_i\|$, $sup_{0\le i\le n}\|x_0 - \xi_i\|\}$

Firstly, we show that c_n= d_n. Assume that $a_n > 0$ there are six cases

Case.1 $a_n = \|x_i - x_j\|$ for some $0 \le i \le j \le n$.

from (2.5) $a_n = \|x_i - x_j\|$

$$\le (1-\alpha_{j-1})\|x_i - \mu_{j-1}\| + \alpha_{j-1}\|x_i - \xi_{j-1}\|$$

$$\le (1-\alpha_{j-1})\|x_i - \mu_{j-1}\| + \alpha_{j-1}c_n$$

$$c_n(1-\alpha_{j-1}) \le (1-\alpha_{j-1})\|x_i - \mu_{j-1}\|$$

Which implies $c_n = \|x_i - \mu_{j-1}\|$ and by induction, $c_n = \|x_i - x_i\| = 0$, contraction with $c_n > 0$ so must be 0. Case.2 $c_n = \|x_i - \mu_{j-1}\|$, for some $0 \le i \le j \le n$ .then from (2.5) and condition (2.6)

$$c_n = \|x_i - \mu_{j-1}\| \le (1-\alpha_{i-1})\|\mu_{i-1} - \mu_j\| + \alpha_{i-1}\|\xi_{i-1} - \mu_j\|$$

$$\le (1-\alpha_{i-1})\|\mu_{i-1} - \mu_j\| + \alpha_{i-1}\, D(Fy_{i-1}, Fx_j)$$

$$\le (1-\alpha_{i-1})\|\mu_{i-1} - \mu_j\| + \alpha_{i-1}qc_n,\ q < 1$$

$c_n \le \|\mu_{i-1} - \mu_j\|$ then $c_n = \|\mu_{i-1} - \mu_j\|$

and by induction,$\|x_0 - \mu_j\| = c_n$

Case.3$c_n = \|x_i - y_i\|$ for some $0 \le i \le j \le n$

$$c_n = \|x_i - y_i\| \le \beta_j\|x_i - \mu_j\| + (1-\beta_j)\|x_j - \mu_j\|$$

This implies that:

$c_n = \|x_j - \mu_j\|$ or $c_n = \|x_i - \mu_j\|$

by case. 2

$$c_n = \|x_0 - \mu_j\|$$

Case.4 $c_n = \|x_i - \xi_j\|$ by similar way to case.2

Case.5 $c_n = \|y_i - \mu_j\|$

Case.6 $c_n = \|y_i - \xi_j\|$

The remaining case of c_n are:

$c_n = \|\mu_i - \mu_j\|$or $c_n = \|\mu_i - \xi_j\|$

Are impossible and for any set A, denote

δ(A)=dim (A).

We show that $d_n = \max\{\delta_n, \varepsilon_n\}$ where $\delta_n = \delta(o(x_0, n))$ and $\varepsilon_n \delta(o(y_0, n))$

Suppose that

$d_n = \delta_n = \|x_0 - \mu_i\|$ for some $0 \leq i \leq n$ then $\delta < \infty$

If $i > 0$ then from (2.6)

$$\delta_n = \|x_0 - \mu_i\| \leq \|x_0 - \mu_1\| + \|\mu_1 - \mu_i\|$$

$$\leq \|x_0 - \mu_1\| + D(Fx_0, Fx_i)$$

$$\leq \|x_0 - \mu_1\| + q\delta_n$$

$$\delta_n \leq \frac{1}{1-q}\|x_0 - \mu_1\|$$

Similarly, we can show that $\varepsilon_n \leq \frac{1}{1-q}\|x_0 - \mu_1\|$ which is complete the proof.

Proof of Theorem (3.1)

For each $n \geq 0$ difine

$A_n = \{\langle x_i\rangle \cup \langle y_i\rangle \cup \langle \mu_i\rangle \cup \langle \xi_i\rangle$, Where$0 \leq i \leq n\}$

By using the same argument of proof lemma (3.2), we can show that

$$r_n := diam(A_n) = \max\{sup_{j\geq n}\|x_n - \mu_j\|, sup_{j\geq n}\|x_n - \xi_j\|\}$$

By using (2.5)

$$\|x_n - z\| \leq (1 - \alpha_{n-1})\|\mu_{n-1} - z\| + \alpha_{n-1}\|\xi_{n-1} - z\|$$

$$\leq (1 - \alpha_{n-1})r_{n-1} + \alpha_{n-1}H(Fy_{n-1}, Fz)$$

$\leq (1-\alpha_{n-1})r_{n-1} + q\alpha_{n-1}r_{n-1}$

for each n, assume that $r_n > 0$, then it follows that:

$r_n \leq (1-\alpha_{n-1})r_{n-1} + q\,\alpha_{n-1}\,r_{n-1}$

$$r_{n-1} - r_n \geq (1-q)\alpha_{n-1}\,r_{n-1} \tag{3.1}$$

This implies $\langle r_n \rangle$ is none increasing in n,

Therefore, there exist r such that$r = \lim_{n\to\infty} r_n$.

Suppose that $r > 0$. from (3.1)

$(1-q)\alpha_{n-1}r \leq (1-q)\alpha_{n-1}r_{n-1} \leq r_{n-1} - r_n$

or

$$(1-q)r\,\sum_{k=0}^{n}\alpha_k \leq \sum_{k=0}^{n}(r_k - r_{k+1}) = r_0 - r_{n-1} \tag{3.2}$$

when $n \to \infty$, the right hand side of (3.2) is bounded but the hypothesis of$\langle \alpha_n \rangle$, makes the left hand side is un bounded which is contradiction. so, r = 0. Hence $x_n \to z\ as\ n \to \infty$

Which is complete this proof.

As application of theorem (3.1) we can prove a fixed point result for Contraction of integral type of summable$\mu: [0,+\infty) \to [0,+\infty)$ (i.e. with finite

Integral on each compact subset of $[0,+\infty)$:

Theorem (3.3): let M be a nonempty closed convex subset of Banach space X and $F: M \to M$ be an operater satisfying the following condition:

$$\int_o^{d(Fx,Fy)} \mu(t)dt \leq q \int_o^{\max\{\|x-y\|,\|x-Fx\|,\|y-Fy\|,\|x-Fy\|,\|y-Fx\|,\ \|F^2x-x\|,\|F^2x-Fx\|,\|F^2x-y\|,\|F^2x_{Fy}\|\}} \mu(t)dt$$

for all $x, y \in$X and $0 < q < 1$, where $\mu: [0,+\infty) \to [0,+\infty)$ is a Lebesgue-integrablesummable mapping and for each $\varepsilon > \int_0^{\varepsilon}\mu(t)dt > 0$. Let $\langle x_n \rangle$be defined by the iteration (2.5) with$\sum_0^{\infty}\alpha_n\beta_n = \infty$, then$\langle x_n \rangle$converges strongly to the unique fixed point of F.

Proof: by taking $\mu(t) = 1$the proof of theorem (3.3) is follows from theorem (3.1)over $[0,+\infty)$because the result of the integral summable mappings satisfying condition (2.6) and it is just transforms in to a g.q.m.c-mappings not involving integral. This completes the proof.

Lemma (3.4) [10]: let $\langle b_n \rangle$ be a non negative sequence where $\lambda_n \in (0,1)$ for all $n \geq n_0, \sigma_n = o(\lambda_n)$ and $\sum_{n=1}^{\infty}\lambda_n = \infty$. This is satisfying the following inequality:

$b_{n+1} \leq (1-\lambda_n)b_n + \sigma_n$, then $\lim_{n\to\infty} b_n = 0$.

Theorem (3.5): let $F: M \to CB(M)$be multi valued mappings satisfying condition (2.6), let $\alpha_n > 0$ for all $n \geq 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$

Then for$u_0 = x_0 \in M$, the following are equivalent:

- The $P_n M_n$ (2.2) converges to z ;
- The s-iteration (2.5) converges to$z$.

Proof: By theorem (3.1), F has a fixed point, say, z and the sequence $\langle x_n \rangle, \langle y_n \rangle, \langle \mu_n \rangle, \langle \xi_n \rangle$ are bounded. Similarly, the sequence $\langle u_n \rangle, \langle \theta_n \rangle$ also are bounded. In order to prove the equivalence between (2.2) and (2.5), we need to prove that

$$\lim_{n\to\infty} \|x_n - u_n\| = 0 \tag{3.2}$$

set, $r_n = \max\{sup_{j\geq n}(\|x_n - \mu_n\|) \cup sup_{j\geq n}(\|x_n - \theta_n\|) \cup sup_{j\geq n}(\|x_n - \xi_n\|) \cup sup_{j\geq n}(\|u_n - \mu_n\|) \cup sup_{j\geq n}(\|u_n - \theta_n\|) \cup sup_{j\geq n}(\|u_n - \xi_n\|)\}$

Then the following are true:

- by using (2.5)

$$\|x_n - \mu_j\| \leq (1 - \alpha_{n-1})\|\mu_{n-1} - \mu_j\| + \alpha_{n-1}\|\xi_{n-1} - \mu_j\|$$

$$\leq (1 - \alpha_{n-1})D(Fx_{n-1}, Fx_j) + \alpha_{n-1}D(Fy_{n-1}, Fx_j)$$

$$\leq (1 - \alpha_{n-1})qr_{n-1} + \alpha_{n-1}qr_{n-1}$$

$$r_n \leq qr_{n-1} \tag{3.3}$$

$$\|x_n - \theta_j\| \leq (1 - \alpha_{n-1})\|\mu_{n-1} - \theta_j\| + \alpha_{n-1}\|\xi_{n-1} - \theta_j\|$$

$$\leq (1 - \alpha_{n-1})D(Fx_{n-1}, Fv_j) + \alpha_{n-1}D(Fy_{n-1}, Fv_j)$$

$$\leq (1 - \alpha_{n-1})qr_{n-1} + \alpha_{n-1}qr_{n-1}$$

$$r_n \leq qr_{n-1} \tag{3.4}$$

$$\|x_n - \xi_j\| \leq (1 - \alpha_{n-1})\|\mu_{n-1} - \xi_j\| + \alpha_{n-1}\|\xi_{n-1} - \xi_j\|$$

$$\leq (1 - \alpha_{n-1})D(Fx_{n-1}, Fy_j) + \alpha_{n-1}D(Fy_{n-1}, Fy_j)$$

$$\leq (1 - \alpha_{n-1})qr_{n-1} + \alpha_{n-1}qr_{n-1}$$

$$r_n \leq qr_{n-1} \tag{3.5}$$

$$\|u_n - \mu_j\| \leq \|\theta_{n-1} - \mu_j\|$$

$\leq D(Fv_{n-1}, Fx_j)$

$$r_n \leq \; qr_{n-1} \tag{3.6}$$

$\|u_n - \theta_j\| \leq \|\theta_{n-1} - \theta_j\|$

$\leq D(Fv_{n-1}, Fv_j)$

$$r_n \leq \; qr_{n-1} \tag{3.7}$$

$\|u_n - \xi_j\| \leq \|\theta_{n-1} - \xi_j\|$

$\leq D(Fv_{n-1}, Fy_j$

$$r_n \leq \; qr_{n-1} \tag{3.8}$$

It is clear that all (3.3), (3.4), (3.5), (3.6), (3.7), (3.8) and using theorem (3.1) that the sequence $\langle r_n \rangle$ is non increasing in $n$and positive. There exist r such that

$$\lim_{n\to\infty} r_n = r, r = 0$$

Therefore

$$\lim_{n\to\infty} \|x_n - \mu_n\| = 0, \lim_{n\to\infty} \|x_n - \theta_n\| = 0,$$

$$\lim_{n\to\infty} \|x_n - \xi_n\| = 0, \lim_{n\to\infty} \|u_n - \mu_n\| = 0, \tag{3.9}$$

$$\lim_{n\to\infty} \|u_n - \theta_n\| = 0, \lim_{n\to\infty} \|u_n - \xi_n\| = 0.$$

Suppose now that the s- iteration converges, then one has

$\|x_{n+1} - u_{n+1}\| \leq (1 - \alpha_n)\|\mu_n - \theta_n\| + \alpha_n\|\xi_n - \theta_n\|$

$\leq (1 - \alpha_n)(\|\mu_n - x_n\| + \|x_n - u_n\| + \|u_n - \theta_n\|)$

$+\alpha_n(\|\xi_n - x_n\| + \|x_n - \theta_n\|)$

$$\|x_{n+1} - u_{n+1}\| \leq (1 - \alpha_n)\|x_n - u_n\| + \alpha_n(\|\xi_n - x_n\| + \|x_n - \theta_n\|) \tag{3.10}$$

Using (3.9) and (3.10) and lemma (3.4), with

$\lambda_n \coloneqq \|x_n - u_n\|,$

$\sigma_n \coloneqq \alpha_n(\|\xi_n - x_n\| + \|x_n - \theta_n\|),$

$\sigma_{n=}o(\alpha_n)$,

We have $\lim_{n\to\infty}\lambda_n = 0$, that is, (3.2) holds. The relation

$$\|u_n - z\| \leq \|x_n - u_n\| + \|x_n - z\| \to 0$$

Then the $P_nM_n$iteration converges too. Suppose now that the P_nM_n iteration converges, then one has

$$\|u_{n+1} - x_{n+1}\| \leq (1-\alpha_n)\|\theta_n - \mu_n\| + \alpha_n\|\theta_n - \xi_n\|$$

$$\leq (1-\alpha_n)(\|\theta_n - u_n\| + \|u_n - x_n\| + \|x_n - \mu_n\|)$$

$$+\alpha_n(\|\theta_n - u_n\| + \|u_n - \xi_n\|)$$

$$\|u_{n+1} - x_{n+1}\| \leq (1-\alpha_n)\|u_n - x_n\| + \alpha_n(\|\theta_n - u_n\| + \|u_n - \xi_n\|) \quad (3.11)$$

Using (3.9) and (3.11) and lemma (3.4), with $\lambda_n \coloneqq \|u_n - x_n\|$

$$\sigma_n \coloneqq \alpha_n(\|\theta_n - u_n\| + \|u_n - \xi_n\|)$$

$$\sigma_n = o(\alpha_n),$$

We have$\lim_{n\to\infty}\lambda_n = 0$, that is, (3.2) holds. The relation

$\|x_n - z\| \leq \|u_n - x_n\| + \|u_n - z\| \to 0$. Then the s-iteration converges too.

Which is complete the proof.

As an application of theorem (3.5)

Example (3.6)

Let $f: [0:8) \to [0:8)$ defined by $f(x) = \frac{x^3+9}{10}$. Then f is an increasing function. By taking $\beta_n = \alpha_n = \frac{1}{(1+n)^{\frac{1}{2}}}$, with fixed point=1and initial points: $u_0 = x_0 = 0.6$. In this example we using Mat lap to see that s-iteration equivalent with Picard-Mann iteration listed in Table 1.

Table 1

n	S- iteration	Picard-Mann
1	0.978275778969600	0.978275778969600
2	0.995851011484343	0.996773763535413
3	0.999049483160993	0.999424935574572
4	0.999764960104952	0.999887919215415
5	0.999939373359160	0.999976904099156
6	0.999983934877438	0.999995051392005
7	0.999995662484216	0.999998908205991
8	0.999998812610045	0.999999753523607
9	0.999999671489158	0.999999943310442
10	0.999999908345504	0.999999986757764
..	…	…
22	0.999999999999972	0.999999999999999
23	0.999999999999992	1

Table 1 – Cond.,

24	0.999999999999998	1
25	0.999999999999999	1
26	1	1

Remark (3.7): Let $F: M \to CB(M)$ be satisfying condition (2.6) then, it is not difficult to show that:

$$D(Fx, Fy) \leq \delta\{\|x - y\| + d(x, Fx) + d(x, F^2x\} \quad (3.12)$$

and $D(Fx, Fy) \leq \delta\{\|x - y\| + d(y, Fx) + d(y, F^2x\}$ (3.13)

for all x, y in M and $\delta = \max\{q, \frac{q}{1-q}\}$.

Theorem (3.8): let $F: M \to CB(M)$ be multi valued mappings satisfying condition (2.6) let $\langle x_n \rangle, \langle u_n \rangle$be the s-iteration and Picard-Mann iteration respectively defined by (2.5) and (2.2) for $x_0, u_0 \in M$ with $\langle \alpha_n \rangle$and $\langle \beta_n \rangle$ real sequences such that $0 \leq \alpha_n, \beta_n \leq 1$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then $\langle x_n \rangle$and $\langle u_n \rangle$ converge to the unique fixed point of F, and moreover, the s-iteration converges faster, than the Picard-Mann iteration, to the fixed point of F.

Proof: by using Remark (3.6) and definition of s-iteration we have

$$\|x_{n+1} - z\| \leq (1 - \alpha_n)\|\mu_n - z\| + \alpha_n\|\xi_n - z\| \quad (3.14)$$

suppose $x = z$ and $y = x_n$, by (3.12) we get

$$\|\mu_n - z\| \leq \delta\|x_n - z\| \quad (3.15)$$

If $x = z$ and $y = y_n$ by (3.12) we get

$$\|\xi_n - z\| \leq \delta\|y_n - z\| \quad (3.16)$$

put (3.14) and (3.15) in (3.14)

$$\|x_{n+1} - z\| \leq (1 - \alpha_n)\delta\|x_n - z\| + \alpha_n\delta\|y_n - z\| \quad (3.17)$$

$$\|y_n - z\| \leq (1 - \beta_n)\|x_n - z\| + \beta_n\|\mu_n - z\|$$

$$\leq (1 - \beta_n)\|x_n - z\| + \beta_n\,\delta\|x_n - z\|$$

$\leq (1 - \beta_n + \beta_n\,\delta)\,\|x_n - z\|$ Put in (3.17)

$$\|x_{n+1} - z\| \leq \{(1 - \alpha_n)\delta + \alpha_n\delta(1 - \beta_n + \beta_n\,\delta)\}\|x_n - z\|$$

$$\leq \{\delta - \alpha_n\beta_n\,\delta + \alpha_n\beta_n\delta^2\}\|x_n - z\|$$

$$\leq \prod_{k=1}^{n}\{\,\delta - \alpha_k\beta_k\,\delta + \alpha_k\beta_k\delta^2\}\|x_0 - z\|$$

let $a_n = \{\delta - \alpha_k\beta_k\,\delta + \alpha_k\beta_k\delta^2\}$

$$= (1 - \alpha_k)\delta + \alpha_k\delta^2$$

Similarly, let $\langle u_n \rangle$ be the Picard-Mann iteration defined by (2.2) then, we have in (3.13) let $x = z, y = v_n$

$$\|u_{n+1} - z\| = \|q_n - z\|$$

$$\leq 3\delta\|v_n - z\| \tag{3.18}$$

$$\|v_n - z\| \leq (1 - \alpha_n)\|u_n - z\| + \alpha_n\omega_n \tag{3.19}$$

let $y = u_n, x = z$in (3.13), we have

$\|\omega_n - \mathrm{z}\| \leq 3\delta\|u_n - \mathrm{z}\|$ put in (3.19) and put (3.19) in(3.18)

$$\|u_{n+1} - z\| \leq 3\delta\{(1 - \alpha_n)\|u_n - z\| + \alpha_n 3\delta\|u_n - z\|\}$$

$$\leq \{3\delta - 3\alpha_n\delta + 9\alpha_n\delta^2\}\|u_n - z\|$$

$$\leq \prod_{k=1}^{n}\{3\delta - 3\alpha_k\delta + 9\alpha_k\delta^2\}\|u_0 - z\|$$

let $b_n = 3\delta - 3\alpha_k\delta + 9\alpha_k\delta^2$

$$= (1 - \alpha_k)3\delta + \alpha_k(3\delta)^2$$

By using the Definition (2.1) we first note that $a_n < b_n$ for each k and

$\frac{a_n}{b_n} = \frac{(1-\alpha_k)\delta+\alpha_k\delta^2}{(1-\alpha_k)3\delta+\alpha_k(3\delta)^2}$, since$(1 - \alpha_k)\delta + \alpha_k\delta^2 < (1 - \alpha_k)3\delta + \alpha_k(3\delta)^2$

Now, for each k, we know that

$\frac{\min\{(1-\alpha_k)\delta+\alpha_k\delta^2\}}{\max\{(1-\alpha_k)3\delta+\alpha_k(3\delta)^2\}} < 1$ and $(\frac{\min\{(1-\alpha_k)\delta+\alpha_k\delta^2\}}{\max\{(1-\alpha_k)3\delta+\alpha_k(3\delta)^2\}}) \to 0$

Clearly $\prod_{k=1}^{n}\frac{(1-\alpha_k)\delta+\alpha_k\delta^2}{(1-\alpha_k)3\delta+\alpha_k(3\delta)^2} < (\frac{\min\{(1-\alpha_k)\delta+\alpha_k\delta^2\}}{\max\{(1-\alpha_k)3\delta+\alpha_k(3\delta)^2\}})^n$

and $\lim_{n\to\infty}\frac{a_n}{b_n} = 0$, as $n \to \infty$

As a converge sequence we obtain that.

Example

Let $\mathrm{f}: [0,1] \to [0,1]$ be defined by $\mathrm{f(x)} = (1 - \mathrm{x})^9$ then f is a decreasing function. The comparison of the convergence for S- iteration and Picard Mann is shown where the initial points: $\mathrm{u}_0 = \mathrm{x}_0 = 0.6$ and $\beta_n = \alpha_n = \frac{1}{(1+n)^{\frac{1}{4}}}$, where the fixed z = 0.175699 is listed in Table 2. we see that the S-iteration converges faster than Picard-Mann.

Table 2

n	S- iteration	Picard-Mann
1	0.997643	0.997643
2	0.177487	0.211069
3	0.178056	0.255567
4	0.177781	0.302197
5	0.176920	0.296542
6	0.176148	0.273882
7	0.175788	0.257326
8	0.175704	0.243684
9	0.175699	0.232251
10	0.175699	0.222550
11	0.175699	0.214259
12	0.175699	0.207153
..	…	…
31	0.175699	0.175704
32	0.175699	0.175701
33	0.175699	0.175700
34	0.175699	0.175699
35	0.175699	0.175699

REFERENCES

1. Ciric, L. B. A generalization of Banach's contraction principle, Proc. Amer. Math. Soc. Vol. 45, pp. 267-273. (1974).
2. Nadler, N. B. Multi-valued contraction mapping, Pacific J. Math. Vol. 30, no. 2, pp. 475-488.(1969)
3. Kannan, R. some results on fixed points, Bull. Calcutta Math. Soc. Vol. 60, pp. 71-76. (1968).
4. Chatterjee, S. K. Fixed point theorems compacts, Rend. Acad. Bulgare Sci. Vol. 25, pp. 727-730.(1972).
5. Zamfirescu, T. fixed point theorems in metric space, Arch. Math. Vol. 23, pp. 292-298. (1972).
6. Babu, G. V. Prasad, K. N, "Mann iteration converges faster than Ishikawa iteration for the class of Zamfirescu operators ", fixed point theory and applications, pp. 1-6.(2006)
7. Berinde, V. "Picard iteration converges faster than Mann iteration for a class of quasi-contractive operators", fixed point theory and applications, Vol.1, pp.1-9.(2004)
8. Berinde, V. "on the convergence of the Ishikawa iteration in the class of quase-contractive operator", Acta Math. Univ. Comeniaca, Vol.73 no.1, pp. 119-126. (2004).
9. Berinde, V. "iterative approximation of fixed points" Springer, Berlin, V. Heidelberg (2007)
10. Ciric, L. B. "convergence theorems for a sequence of Ishikawa iteration for nonlinear quasi-contractive mappings", Indian J. Pure Appl. Math. Vol.30, no.4, pp. 425-433.(1999)
11. Olaleru, J. O. "A comparison of Picard and Mann iteration for quasi-contractive maps", fixed point theory, Vol.8, no.1, pp. 87-95.(2007)
12. Soltuz, S. M. "the equivalence of Picard, Mann and Ishikawa iterations dealing with quasi-contractive operators", Math. Communications Vol.10, no.1, pp. 81-88.(2005).

13. Rhoades, B. E. A comparison of various definitions of contractive mappings, Trans. Amer. Math. Soc. Vol. 226, pp. 257-290. (1977)

14. Mann, W. R. “mean value methods in iteration”, Proceedings of the American Math. Soc., Vol. 4, no. 3, pp. 506-510. (1953)

15. Ishikawa, S. “fixed point by a new iteration method”, Proceeding of the American Math. Soc., Vol. 44, no. 1, pp. 147-15. (1974)

16. Agarwal, R. P. O’ Regan, O. and Sahu, D. R. “Iterative construction of fixed points of nearly asymptotically non expansive mappings”, Journal of Nonlinear and Convex Analysis, Vol. 8, no. 1, pp. 6179-6189, (2007)

17. Thianwan, S. “common fixed point of new iteration for two asymptotically non expansivenonself -mappings in a Banach space”, Journal of Computational and Applied Math., Vol. 224, no. 2, pp. 688-695, (2009)

18. Khan, SH. A Picard-Mann hybrid iterative process. fixed point theory Appl., doi:10.1186/1687-1812-2013-69. (2013)

19. Gursoy, F, Karakaya, V, “A Picard-S hybrid type iteration method for solving a differential equation with retarded argument”, pp. 16. (2014).

20. Abed, S. S, The convergence of Ishikawa iteration with respect to some types of multi valued mappings, 1 stscie. nati. Conf. for women researches, part 1, pp. 320-337. (2012)

21. Dung, N. V., Kumam, P., Sitthithakerngkiet, K, A Generalization of Ciric fixed point theorems, Facu. of Sci. and Math. Filo. Vol. 29, no.7, pp 1549-1556. (2015)

22. Rhoades, B. E. fixed point iteration using infinite matrices, Trans. Amer. Math. Soc. 196, pp161-176. (1974).

23. Zeidler E, “nonlinear functional analysis and applications (fixed point theorems), Springer Verlag New York Inc. (1986)